A COMPARISON OF THE LIBERMAN-ROSS AND MANN-GRUBBS METHODS

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by

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This paper compares the Lieberman-Ross (LR) method, a statistically exact procedure for computing system reliability bounds, with the Mann-Grubbs (MG) procedure, an approximately optimum method for computing such bounds. For systems with exponentially distributed failure times, it is shown that the MG procedure, using the same data as the LR procedure, will always compute a lower bound than the LR bound. Simulation methods are used to infer that only when failure data are ordered so that a significant portion of the data

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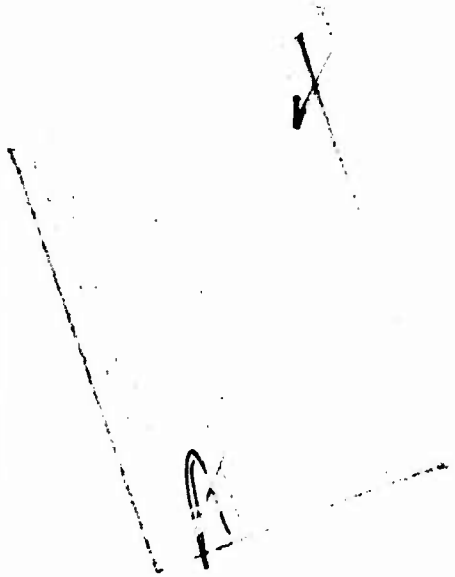
is not incorporated by the LR procedure, will the LR procedure have a reasonable expectation of producing a superior bound to the MG bound. This is interpreted as dictating a data order that discards failure data on the least reliable component samples.



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I. INTRODUCTION

The reliability of any system composed of N components is defined:

$$R(T_o) = \text{Probability } (T \geq T_o)$$

where T denotes the time of first system failure. A significant practical problem is that it is often reasonably easy, and comparatively inexpensive, to determine information about the reliability of separate component types, denoted $R_i(T_o)$, $i = 1,2,\ldots, N$, but fairly difficult and extremely expensive to determine total system reliability directly, since failure testing often destroys the system.

In the most common analytically treated case, statistically independent components in series, where each of the component types has an assumed exponential distribution of failures, i.e.,

$$R_i(T_o) = e^{-\lambda_i T_o}, \; i = 1,2,\ldots, N , \tag{1}$$

the system reliability is given by

$$R(T_o) = e^{-\lambda T_o}, \; \lambda = \sum_{i=1}^{N} \lambda_i . \tag{2}$$

(It should be noted in complex systems that these are valid only when all components operate the same fraction of mission time.)

A statistically exact procedure for obtaining upper bounds on λ from data on component failure, here called the LR procedure, has been developed by Lieberman and Ross (1971). In this procedure k_i items of each type component, with individual failure times T_{ij}, $i = 1,2,\ldots, N$ and $j = 1,2,\ldots, k_i$ are to be observed. A time U is defined:

$$U = \min_i \left\{ \sum_{j=1}^{k_i} T_{ij} \right\} . \tag{3}$$

i.e., U is the cumulative time at which one first exhausts all the components of one type. The number of each type of component which has been used up to this time if given by:

$$n_i = \text{largest } \{j \leq k_i \mid \sum_{\ell=1}^{j} T_{i\ell} \leq U_{min}\}. \tag{4}$$

Then, if

$$K = \sum_{i=1}^{N} n_i \tag{5}$$

(i.e., total components used), Lieberman and Ross showed that $2\lambda U$ follows the χ^2_{2K} distribution; hence upper bounds for λ can be estimated.

A second procedure for estimating upper bounds on λ is the approximately optimum procedure for Type II censoring (fixed number of failures) developed by Mann and Grubbs (1974), which approximates the optimum exact bound of El Mawaziny (1965). In this Mann and Grubbs (hereafter referred to as MG) procedure, it is assumed that testing of all components of type i commences simultaneously, and continues until r_i of the k_i original have failed. (It is permissible that $r_i = k_i$.) Then, if the individual failure times are ordered so that $T_{ij} \leq T_{i(j+1)}$, the total test time for the i^{th} component type is defined,

$$Z_i = \sum_{j=1}^{r_i} T_{ij} + (k_i - r_i)T_{ir_i}. \tag{6}$$

The approximate optimum upper bound for λ, at confidence level α, is given by:

$$\lambda_b^{MG} = m \left\{ 1 - \frac{v}{9m^2} + \frac{\eta_\alpha \sqrt{v}}{3m} \right\}^3, \tag{7}$$

where η_α is the $100\alpha^{th}$ percentile of the standard normal distribution, and

$$m = \sum_{i=1}^{N} \{(r_i - 1)/Z_i\} + 1/U \ , \tag{8}$$

and

$$v = \sum_{i=1}^{N} \{(r_i - 1)/Z_i^2\} + 1/U^2 \ , \tag{9}$$

with U defined by (3) above.

A major drawback of the LR technique is that in using K as defined by (4)-(5), one "discards" the data on the

$$NL = \sum_{i=1}^{N} \{k_i - n_i\} \tag{10}$$

components which do not fail by the time U. While necessary for the statistical validity of the procedure, this means that, effectively, the LR procedure assumes components of each type are tested sequentially, i.e. testing of the $(j+1)^{st}$ component of type i commences with failure of j^{th} component of that type. This is seldom the testing scheme in practice.

An immediate consequence of this and (4) is that the LR procedure is Data-Order Dependent, that is, permuting the second subscript on T_{ij} (i.e., in essence permuting the order in which the failures are observed) can alter n_i, and hence the LR estimated bound. This can lead to significant practical problems, for often failure data on individual components is gathered in simultaneous (parallel) testing of all k_i components, rather than in sequential testing. Thus there is no preferred ordering, and, before using the LR procedure, the analyst is faced with the formidable task of deciding on the "best" way to assign the T_{ij}.

It is obvious that the MG procedure does not suffer these drawbacks. (Note that the data order dependence of the LR method is a mixed blessing, since it may allow the analyst to construct several alternative bounds to λ, all of equal statistical validity, by simply permuting the data.)

The purpose of this paper is to compare the LR and MG procedures in terms of the expected bounds, and effects of "discarded" data, using both analytic and simulation techniques.

II. COMPARISON OF THE TWO METHODS-ANALYTIC RESULTS

Immediate comparison of the LR and MG procedures is complicated by the fact the test schemes are conceptually different. As noted above, the derivation of the LR procedure presupposes that testing of the $(j+1)^{st}$ component of any type begins with the failure of the j^{th} component of that type, and that <u>all</u> component testing ceases at U, with the exhaustion of the supply of one type of component. Thus, conceptually, the LR ends with $(N-1)$ components (one of each of the other types) still undergoing testing, and no failed plus others, $(NL+1-N)$ to be exact, not even tested. By contrast, the MG procedure implicitly assumes, for each i, testing of all k_i components of type i starts simultaneously, and terminates with a failure of some one of them. Thus before data gathered according to the MG procedure can be utilized to produce a LR bound, the analyst must first choose one (or more) a priori orderings of the data.

However, for exponential failure data, we appeal to the constant failure rate property to convert data gathered in a LR test format into MG format. Precisely, constant failure rate implies that irrespective of whether testing of the i^{th} component type terminates at a failure, the sufficient statistics are total test time and number of failures observed in that time. Thus, if all testing is halted at U according to the LR model, then, we shall apply the MG procedure (providing each component type experienced at least one failure) with:

$$\left.\begin{aligned} z_i &= U \\ r_i &= n_i \geq 1 \end{aligned}\right\} i = 1,2,\ldots, N \; . \tag{11}$$

(This does mildly violate the MG assumptions, in that the n_i is random, not set a priori. However, any effect sould be noticeable only in extremely small sample sizes, and will not affect our basic conclusion unless $P\{\lambda_b^{MG} < \lambda\}$ is significantly increased as a result.)

From this, it immediately follows that

$$m = \sum_{i=1}^{N} \{(n_i - 1)/U\} + 1/U$$

$$= \left\{ \sum_{i=1}^{N} (n_i - 1) + 1 \right\} /U = (K + 1 - N)/U, \tag{12}$$

where K is given by (5), and, similarly,

$$\nu = (K + 1 - N)/U^2 . \tag{13}$$

But, with these values, (7) reduces to

$$\lambda_b^{MG} = \frac{(K + 1 - N)}{U} \left\{ 1 - \frac{1}{9(K + 1 - N)} + \frac{\eta_\alpha}{3\sqrt{(K + 1 - N)}} \right\}^3 . \tag{14}$$

Observe (14) is a strictly increasing function of $(K+1-N)$ for U fixed and $\eta_\alpha < 5.65$ (which includes all cases of practical interest). Thus, since $N \geq 2$,

$$\lambda_b^{MG} \leq \frac{(K-1)}{U} \left\{ 1 - \frac{1}{9(K - 1)} + \frac{\eta_\alpha}{3\sqrt{K - 1}} \right\}^3 . \tag{15}$$

Since the bound produced by the LR method from this data is

$$\lambda_b^{LR} = \frac{\chi^2_{2K}(\alpha)}{2U} , \tag{16}$$

we see that, when the MG procedure is applied to the <u>same data actually used in the LR procedure</u>, the computed bounds satisfy,

$$\left(\lambda_b^{MG}/\lambda_b^{LR}\right) \leq \frac{2(K - 1)}{\chi^2_{2K}(\alpha)} \left\{ 1 - \frac{1}{9(K - 1)} + \frac{\eta_\alpha}{3\sqrt{K - 1}} \right\}^3 . \tag{17}$$

In Table 1, we show values of the right hand side of (17) for representative K and α. It is obvious that, for all practical values of K and α,

$$\left(\lambda_b^{MG}/\lambda_b^{LR}\right) < 1.$$

Thus, when both the Lieberman-Ross and Mann-Grubbs procedures are used to compute an upper bound to λ, if the MG procedure uses only the same failure times as are actually used by the LR procedure, the upper bound produced by the Mann Grubbs procedure will be smaller than that produced by Lieberman-Ross.

An immediate extension of the above result is that the LR procedure may produce lower bounds than the MG procedure only in those cases where the LR procedure does not utilize all the data available. Specifically, consider the situation where the LR procedure utilizes failure times on all but one component of each type except the type which exhausts first, which we assume is type $i = 1$. Testing is assumed to have continued on these $(N - 1)$ components until they also failed. Thus:

$$\begin{aligned} n_1 &= k_1, \\ n_i &= k_i - 1 \quad i = 2, \ldots, N, \end{aligned} \tag{18}$$

and

$$\begin{aligned} Z_1 &= U, \\ Z_i &> U, \quad i = 2, \ldots, N. \end{aligned} \tag{19}$$

Let $\varepsilon > 0$ and arbitrarily small, and assume that each component of type $i \geq 2$, on test at $t = U$, fails at $t = U + \varepsilon$. Thus, MG can be applied with:

$$\left.\begin{aligned} r_i &= k_i \qquad i = 1, 2, \ldots, N, \\ Z_1 &= U \\ Z_i(\varepsilon) &= U + \varepsilon, \quad i = 2, \ldots, N. \end{aligned}\right\} \tag{20}$$

Since, for ε arbitrarily small, $Z_i(\varepsilon) < Z_i$, $i = 2, \ldots, N$, we are thus understating the reliability of components of type i, $i \geq 2$. Hence we expecte the bound computed using $Z_i(\varepsilon)$, denoted $\lambda_b^{MG}(\varepsilon)$ to be conservative, i.e.

$$\lambda_b^{MG} < \lambda_b^{MG}(\varepsilon),$$

and, more specifically

$$\lambda_b^{MG} < \lim_{\varepsilon \to 0} \lambda_b^{MG}(\varepsilon). \tag{21}$$

It is easily shown

$$\lim_{\varepsilon \to 0} \lambda_b^{MG}(\varepsilon) = \frac{K}{U} \left\{ 1 - \frac{9}{K} + \frac{\eta_\alpha}{3\sqrt{K}} \right\}^3, \tag{22}$$

with,

$$K = \sum_{i=1}^{N} (r_i - 1) + 1 = \sum_{i=1}^{N} (k_i - 1) + 1 = k_1 + \sum_{i=2}^{N} (k_i - 1),$$

i.e. K is degree of freedom from the LR procedure on this data. Thus, for this data,

$$\left(\lambda_b^{MG} / \lambda_b^{LR}\right) < \frac{2K}{\chi^2_{2K}(\alpha)} \left\{ 1 - \frac{9}{K} + \frac{\eta_\alpha}{3\sqrt{K}} \right\}^3. \tag{23}$$

Computations show that, for $.75 \leq \alpha \leq .995$, and $2 \leq k \leq 120$, the righthand side of (23) does not differ from unity by more than 0.4%, except for $k = 2$ and $\alpha = .995$, when it differs from unity by 0.8%. (Some of

this may be due to minor inaccuracies in the computer routines used.) Thus, we conclude that the Lieberman-Ross method can not produce a significantly lower, upper bound on λ than the Mann Grubbs method, unless individual component failure times are collected separately, and combined into pseudo-system failures in such a way that the LR procedure assumes some of the components were never tested, i.e. the LR method must deliberately discard some of the data in order to compute a better bound than MG.

Since both the Lieberman-Ross and Mann-Grubbs procedures estimate upper bounds to λ, it follows that, unless the confidence level (α) is sufficiently low that computing a false bound is significantly probable, the method producing the lower estimated bound will be superior. Thus, we look for the LR method to be superior to MG when the ordering of the data chosen for the LR method is such that some, probably a significant fraction, of the available data on failure times is not utilized, i.e. when NL, as defined by (10) satisfies $NL > (N-1)$.

III COMPARISON OF THE TWO METHODS - SIMULATION RESULTS

Having determined that the cases where the Lieberman-Ross method should be superior to the Mann-Grubbs procedure should arise among those where the data ordering used for the LR procedure "discards" some of the available failure data, we proceded to investigate this relationship in more detail. We felt that the most useful measure would be the probability that the LR method would produce a superior bound, as a function of the amount of data unused. Since analytic techniques were not able to shed much additional light in this area, we turned to simulation. The simulation described in [4], which allows generation of component failure data for up to twenty individual components of up to six different types, was modified for this purpose. We restricted ourselves to the two component type (N=2) case for simplicity. Four individual runs, each run consisting of 500 iterations, were conducted for each of the combinations of confidence levels, components and failure rates shown in Table 2. In each iteration, independent exponential failure times were generated for all the components specified of each type, and the MG procedure applied to these data, to produce a bound, λ_b^{MG}, at the specified confidence limits. Then the data were ordered into a single LR pseudo-failure sequence. The manner in which individual component failures were generated served to insure this was a suitably random ordering of the failure times. The LR procedure was applied to this sequence, to produce the bound λ_b^{LR}. The two bounds were compared, and the LR procedure deemed to produce the superior bound if either of the following held:

(1) $\lambda \leq \lambda_b^{LR} < \lambda_b^{MG}$,

(2) $\lambda_b^{MG} < \lambda \leq \lambda_b^{LR}$,

(3) $\lambda_b^{MG} < \lambda_b^{LR} < \lambda$.

In each iteration, this result was compared with the percentage of available (i.e. generated) data not used by the LR procedure:

$$PL = \{ \sum_{i=1}^{N} (k_i - n_i) \} / \{ \sum_{\substack{i=1 \\ i \neq f}}^{N} k_i \} , \qquad (24)$$

where f is the index of the component type which was first exhausted i.e. $n_f = k_f$.

At the completion of each 500 iteration run, the percentage of iterations where the data ordering had produced a superior bound by the LR method was computed, plotted, and tabulated as a function of the percentage of lost data, PL. The results, for each run, were given in a function:

$$TOTR(J) = \left\{ \begin{array}{l} \text{The percentage of data orderings} \\ \text{with } .05(J-1) < PL \leq .05\ J \text{ where} \\ \text{the LR method produced a superior} \\ \text{bound.} \end{array} \right. \qquad (25)$$

(Note TOTR(J) was set to -1.0 if no data orderings with $.05(J-1) < PL \leq .05J$ were observed on that run.) The plotted distribution of values of TOTR(J) for runs at $\alpha = .95$ and $\alpha = .80$ are shown at Figures 1 and 2 respectively. Since TOTR(J) can be regarded as an estimate to the probability that the LR method produces a superior bound, given that between $5(J-1)\%$ and $5J\%$ of the data was ignored,

it is obvious that only if the analyst is willing to order the data in such a way that 80% or more of the available data is not used, will the probability that the LR bound is superior likely exceed one half. (The reason for the superiority of the LR procedure in some cases when little data was discarded is that, in these cases, the MG procedure produced a false bound, i.e. $\lambda_b^{MG} < \lambda$.)

From Figures 1 and 2 it appears the LR procedure performs slightly better vis-a-vis the MG procedure at $\alpha = .80$ than at $\alpha = .95$. Essentially, this is the result of the probability that $\lambda_b^{MG} < \lambda$ increasing as α decreases.

Based on our observation that an analyst would have to order LR data in such a way that the amount of discarded data exceeded 80% of that available before the LR could be reasonably expected to out perform MG, the best strategy to do this would be to pick the T_{ij} in decreasing order, i.e.

$$T_{i(j+1)} < T_{ij} , \quad i = 1, 2, \ldots, N .$$

This is precisely equivalent to discarding, as much as possible, the <u>least</u> reliable samples of each component type. This is not, philosophically, a particularly attractive strategy.

IV. SUMMARY AND CONCLUSIONS

In this paper we have examined two different methods for computing bounds on series system reliability, the statistically exact procedure of Lieberman and Ross, and the approximately optimum procedure of Mann and Grubbs. Our analysis showed that, when used with the same data, the MG procedure would always produce a lower upper bound on λ than the LR procedure. Thus, the MG procedure seems preferable, especially at high confidence levels, since the probability of computing a false bound is correspondingly small there. Further analysis and simulation results showed that the LR procedure could be reasonably expected to produce a superior bound to MG only where the failure times were ordered such that the LR procedure essentially "discarded" most of the data, and this is usually equivalent to retaining samples of only the most reliable components.

Thus, it appears, at present, that using the LR procedure would be preferable to using the MG procedure only in very special circumstances. The only qualification we should note is that we have not fully utilized the LR procedure's ability to usually produce multiple bounds from the same data. This may be exploited in some way to make the LR procedure more competitive, however we are not certain of that at this time.

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TABLE 1

K	$\phi(K)$ 1/						
	$\alpha = .75$	$\alpha = .80$	$\alpha = .85$	$\alpha = .90$	$\alpha = .95$	$\alpha = .99$	$\alpha = .995$
2	0.513	0.534	0.558	0.586	0.625	0.694	0.720
3	0.685	0.698	0.711	0.728	0.751	0.792	0.805
4	0.766	0.774	0.784	0.795	0.810	0.839	0.849
5	0.813	0.819	0.826	0.834	0.846	0.868	0.876
6	0.845	0.849	0.854	0.861	0.870	0.885	0.894
7	0.867	0.870	0.875	0.880	0.887	0.900	0.904
8	0.883	0.886	0.890	0.894	0.900	0.911	0.914
9	0.896	0.899	0.902	0.905	0.910	0.919	0.922
10	0.906	0.909	0.911	0.914	0.919	0.927	0.929
15	0.937	0.938	0.940	0.941	0.944	0.949	0.950
20	0.953	0.954	0.955	0.956	0.957	0.960	0.961
25	0.962	0.963	0.963	0.964	0.965	0.968	0.968
30	0.968	0.969	0.969	0.970	0.971	0.973	0.973
40	0.976	0.976	0.977	0.977	0.978	0.979	0.979

1/ $$\phi(K) = \frac{2(K-1)}{\chi^2_{2K}(\alpha)} \left\{ 1 - \frac{1}{9(K-1)} + \frac{\eta_\alpha}{3\sqrt{K-1}} \right\}^3$$ (Ref. eqn (17))

TABLE 2

A. Sample Sizes Used in Simulation

k_1	k_2
2	2
2	3
2	4
2	5
3	3
3	4
3	5
5	5
5	10
10	10

B. Ratio of Failure Rates Used in Simulation[1]

(λ_1/λ_2) = 10.00, 5.00, 2.50, 1.25, 1.00, 0.500, 0.250, 0.125, 0.100

[1]For the two component case, (λ_1/λ_2) is only rate needed.

LIST OF FIGURES

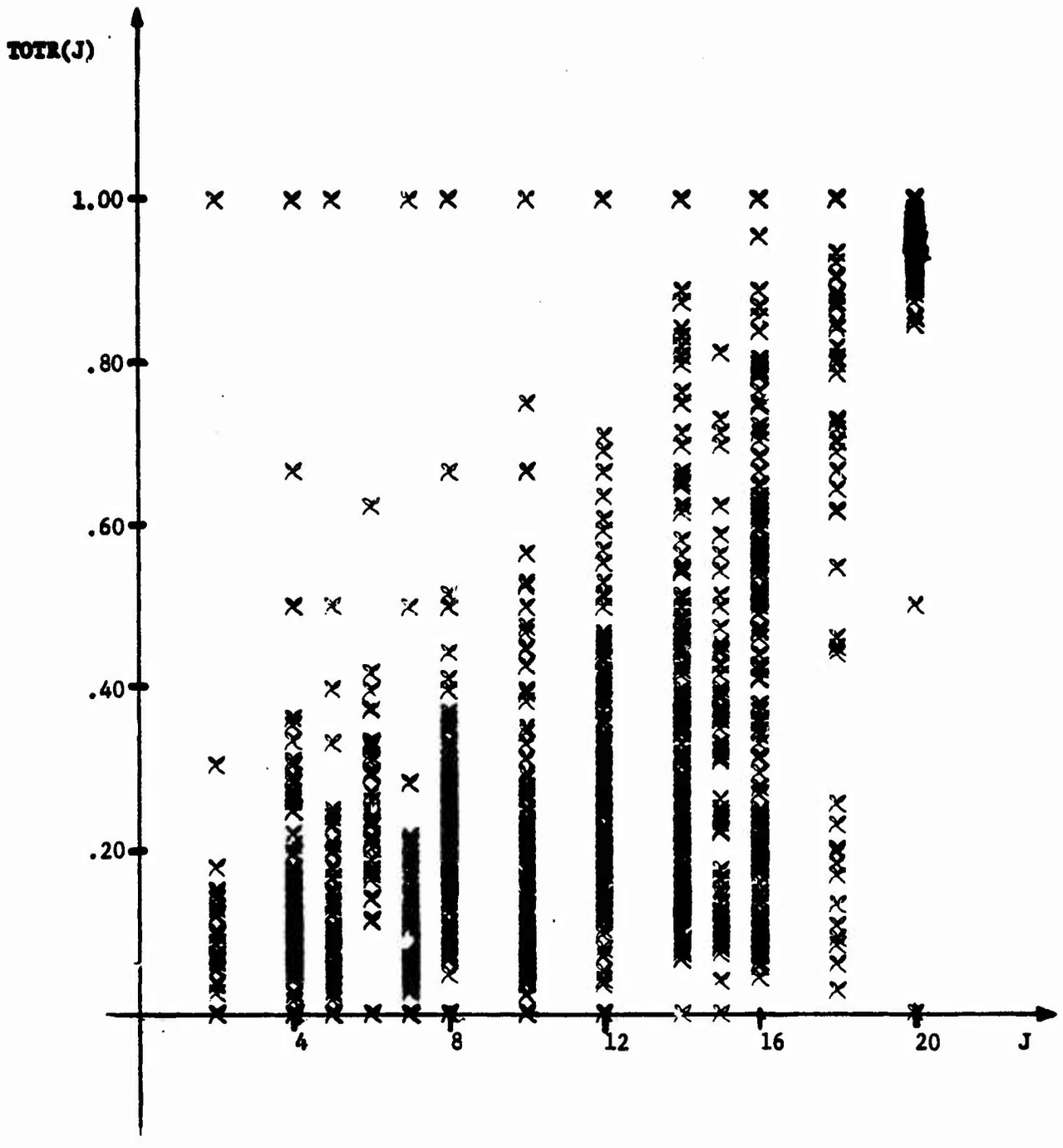

FIGURE 1

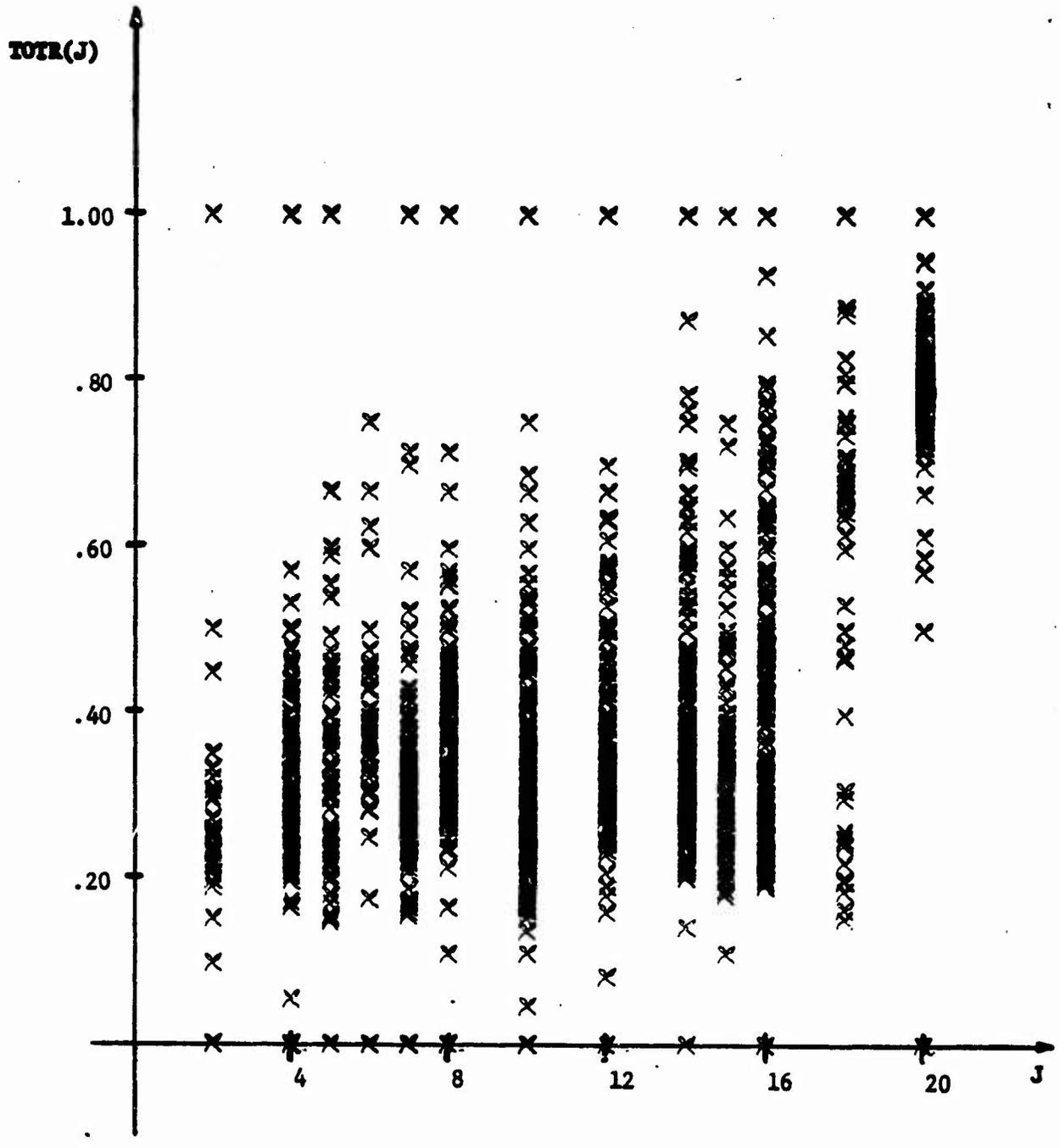

FIGURE 2